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THE ANALYSIS OF IONOSPHERIC H'(F) RECORDS USING THE METHOD OF LEAST SQUARES

by

D. J. Brown

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ABSTRACT

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Electron density-height profiles in the ionosphere can be obtained from virtual height-frequency records by solving an integral equation. A new method of solution is presented, using an iterative minimization scheme where the independent variable is the true height, h, rather than the plasma frequency, f_N. This has the advantage that the method is not restricted to monotonic functions, $f_N(h)$, and can be used for distributions with a "valley". In addition, other height varying parameters can be used, such as the gyrofrequency, whose variations with altitude is important for the upper F region, above the electron peak.

An improved solution is given for the case where no data are available below a certain frequency, f_{min}. This is more general than most available techniques, since no particular model need be assumed for the underlying ionization.

The uniqueness of the solutions which combine data from the ordinary and extraordinary rays for limited frequency ranges is discussed, and suggestions are given for the solution of the valley problem.

Author

I. INTRODUCTION

A. General Statement of the Problem

A widely used method of investigating the structure of the ionosphere consists of transmitting a radio pulse of mean frequency, f, vertically upwards, which is received again on the ground after reflection in the ionosphere. One measures the time interval, τ, between the transmitted and received pulses and defines the quantity "virtual height", h', where $h' \equiv \frac{c\,\tau}{2}$, and c is the free space velocity of light. If the time interval, τ, is measured for a continuous range of frequencies, then one may plot the virtual height, h', as a function of f. The resultant curve is termed an "h'-f curve" or ionogram. We shall only be concerned with frequencies ranging from one to twenty-five megacycles per second, and shall consequently adopt the approach of geometrical ray-optics. Rydbeck (1942a) has investigated the validity of ray-optics and shown that the difference between ray optics and a full wave treatment is of little practical importance for the regular ionospheric layers (the normal E and F regions). We shall assume that the influence of collisions may be neglected, which is permissible if the collision frequency is much less than the wave frequency, i.e. $f \geq 1.0$ mc for heights above 150 km. τ is given by:

$$\tau = 2 \int_0^{h_R} \frac{dh}{V_g} \rightarrow h' = \int_0^{h_R} \frac{c}{V_g}\, dh ,$$

where h is the height measured vertically from the earth's surface, h_R is the height of reflection, and V_g is the group velocity.

If $\mu' \equiv \frac{c}{V_g}$, where μ' is the group refractive index,

$$h' = \int_0^{h_R} \mu' \, dh$$

The expression for μ' has been derived by Appleton and Hartree, who have shown that μ' is a function of the plasma frequency, $f_N(h)$, the gyro-frequency, $f_H(h)$, the dip angle, θ, and the wave frequency, f.

$$h'(f) = \int_0^{h_R} \mu' \, (f_N(h),\ f_H(h),\ \theta,\ f) \, dh$$

Since the ionosphere is an anisotropic medium, a radio pulse which impinges vertically on the ionosphere is split into two components, with different polarizations and group velocities, which we shall call the ordinary and extraordinary modes, i. e.

$$h'_o(f) = \int_0^{h_{R_o}} \mu'_o \, (f_N(h),\ f_H(h),\ \theta,\ f) \, dh \qquad (1.1)$$

$$h'_x(f) = \int_0^{h_{R_x}} \mu'_x \, (f_N(h),\ f_H(h),\ \theta,\ f) \, dh \qquad (1.2)$$

where the ordinary mode is designated by the subscript o and the extraordinary mode by the subscript x.

Equation (1.1) and equation (1.2) are non-linear singular Volterra integral equations of the first kind. The problem is to determine under what conditions equation (1.1) and equation (1.2) uniquely specify the plasma frequency as a function of height, $f_N(h)$; and, providing that these restrictions are satisfied to determine what numerical techniques, which are compatible with reasonable physical assumptions, are available for inverting these equations.

B. Origin and Importance of the Problem

Many countries of the world maintain a number of field stations which have been engaged for many years in taking h'-f records at intervals of one hour or less. Recently a satellite has been orbited which takes h'-f records of the "topside" of the ionosphere. The determination of electron densities from h'-f records is important because of the relative ease with which ionograms are obtained and their low cost as compared to direct probe methods for comparable amounts of data.

C. Previous Solutions

Equation (1.1) and equation (1.2) were derived in the 1930's and since then a number of techniques have been devised for inverting these equations. Several excellent summaries of these methods have been compiled, among them Schmerling (1957) and Thomas (1957). We shall therefore neglect the details of the various methods, but shall concentrate our attention on the assumptions, the rationale behind the assumptions, and the limitations of the assumptions in several of the more successful recent techniques. Since the equations are non-linear in plasma frequency, one naturally seeks physically reasonable assumptions which allow one to linearize the equations, and, if possible, invert them in closed form.

For a large number of cases, i.e. the majority of ionograms obtained at night, the ionosphere may be approximated by a single layer. The plasma frequency, $f_N(h)$, is then a monotonic function of height, and the inverse function $h(f_N)$ exists. One can now consider the equivalent problem of determining $\frac{dh}{df_N}$ from the system:

$$h'_o f = \int_o^{f_{R_o}} \mu'_o (f_N, f_H(h), \theta, f) \frac{dh}{df_N} df_N \qquad (1.3)$$

$$h'_x(f) = \int_o^{f_{R_x}} \mu'_x (f_N, f_H(h), \theta, f) \frac{dh}{df_N} df_N \qquad (1.4)$$

We now have a system of equations linear in $\frac{dh}{df_N}$, but the kernels μ'_o and μ'_x are still too complicated to invert the equations directly. There are, however, a few special cases where the kernel simplifies and the equations can be analytically inverted.

If the earth's magnetic field is neglected, the ionosphere loses its bi-refringent property and only one mode of propagation, the ordinary, is possible, where $\mu'_o = \sqrt{1 - f_N^2/f^2}$

$$h_o(f) = \int_o^{f_{R_o}} \frac{dh/df_N}{\sqrt{1-f_N^2/f^2}} df_N \qquad (1.5)$$

where the plasma frequency of reflection, f_{R_o}, is given by the pole of μ'_o, and $f_{R_o} = f$. Appleton (1930) and de Groot (1930) showed that equation (1.5) may be inverted, since this is merely a form of Abel's integral equation. The solution is given by

$$h(f_N) = \frac{2}{\pi} \int_o^{f_N} \frac{h'(f)\, df}{\sqrt{f_N^2 - f^2}} \qquad (1.6)$$

This also holds for the ordinary ray at the magnetic equator, but is a poor approximation elsewhere on the earth as shown by Shinn and Whale (1952).

For propagation along a magnetic field line (magnetic poles), we have $\mu_o = (1 - f_N^2/f_R^2)^{\frac{1}{2}}$ and $\mu_x = (1 - f_N^2/\tilde{f}_R^2)^{\frac{1}{2}}$. In general[1] $\mu_o' = \frac{\partial}{\partial f} (f\,\mu_o)$ and $\mu_x' = \frac{\partial}{\partial f} (f\,\mu_x)$. Where $f_R^2 = f(f + f_H)$ and $\tilde{f}_R^2 = f(f - f_H)$.

$$h_o'(f) = \int_o^{f_R} \frac{\partial}{\partial f} \left(f \left[1 - f_N^2/f_R^2 \right]^{\frac{1}{2}} \right) \frac{dh}{df_N} \, df_N \qquad (1.7)$$

$$h_x'(f) = \int_o^{\tilde{f}_R} \frac{\partial}{\partial f} \left(f \left[1 - f_N^2/\tilde{f}_R^2 \right]^{\frac{1}{2}} \right) \frac{dh}{df_N} \, df_N \qquad (1.8)$$

Rydbeck (1942b) has shown how equation (1.7) and equation (1.8) may be inverted analytically.

Another approximation is usually made, which is less stringent than either of the previous two approximations. This is that $f_H(h)$ is a constant equal to the value of f_H at some suitably chosen height (usually 100 km). For bottomside soundings where the height range to be considered is roughly 100 km to 300 km, in which range the gyro-frequency, f_H, varies by only 3%, this assumption is quite good. But for topside soundings, where the height range is typically 1000 km to 300 km, the variation is about 20% and the assumption is poor. One now has linear integral equations whose kernels, μ_o' and μ_x' , are

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[1] Budden, K. G. Radio Waves in the Ionosphere, Cambridge University Press, 1961.

complicated yet known functions of f_N^2, f, f_H, and θ. Due to the complexity of the kernels one cannot invert the integral equations directly, but one can derive an infinite set of linear equations whose unknowns, c_i, satisfy the following

$$\frac{dh}{df_N} = \sum_{i=o}^{\infty} c_i \ \varphi \ (f_N)^i \tag{1.9}$$

where ϕ is an arbitrary function of f_N, e.g. f_N, f_N^2 or log f_N. This is a standard technique for solving linear integral equations.[1] This approach has been adopted by Titheridge (1961) and Unz (1961). A similar attack, in that one reduces the linear integral equations to a system of linear algebraic equations, has been given by Budden (1954), Paul (1960), Doupnik (1963), and Paul and Wright (1963). In their approach, $\frac{dh}{df_N}$ is approximated over successive intervals of plasma frequency, as compared to the former approach where one approximation is used for the complete layer. Both, of course, approximate the infinite set of equations by a finite set of equations.

The above methods have been singled out because they are in wide use at the present. They all implicitly or explicitly make the following assumptions:

(a) $f_N(h)$ is a monotonic function of h

(b) $f_H(h)$ is constant.

(c) If $h_o'(f)$ and $h_x'(f)$ are unobserved for $f < f_{min}$, $h(f_N)$ can still be uniquely determined for $f \geq f_{min}$ by judiciously

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[1]Kunz, K. S. Numerical Analysis, McGraw-Hill, 1957.

combining ordinary and extraordinary virtual heights above f_{min}.

(d) If $f_N(h)$ is a non-monotonic function of h, one can determine the monotonic distribution over certain ranges of heights by combining the ordinary and extraordinary virtual heights above f_{min} in the manner used in (c).

D. Specific Statement of the Problem

The conditions for a unique specification of $f_N(h)$ (which is proportional to $\sqrt{\text{electron density}}$) are to be determined. Under these conditions a method is to be developed for obtaining electron density profiles from ionograms.

This method consists of minimizing the squared differences between the observed and calculated virtual heights for a given model.

This method should be applicable to "topside" and "bottomside" ionograms and, consequently, provision must be made for non-monotonic and monotonic electron density distributions, as well as the variation of gyro-frequency with height.

II. THEORETICAL CONSIDERATIONS AND DISCUSSION

A. Conditions Necessary for Uniqueness

Because of equipment limitations, ground based stations do not record virtual heights for frequencies below one or two megacycles per second. The "start" or "low-frequency cut-off" problem consists of inverting the given system of integral equations when $h'_o(f)$ and $h'_x(f)$ are unobserved for f less than some minimum frequency, f_{min}. If more than one layer is present, we say that the electron density profile has a "valley". A "valley" is present whenever $f_N(h)$ is a non-monotonic function of h.

Let us assume initially that $f_{min} = o$, a condition never met in practice. It should be noted that, if f_{min} is quite small, then the $h'_o(f)$ and $h'_x(f)$ curves may be extrapolated from f_{min} to f equals zero and the standard reduction techniques can be applied. The following argument concerns itself with the many instances where extrapolation of the h'-f curves is an uncertain and risky business. Therefore, if $f_{min} = o$ or h'-f curves can be extrapolated and $f_N(h)$ is a monotonic function of h, then either equation (1.3) or equation (1.4) uniquely determines $\frac{dh}{df_N}$[1], and hence $f_N(h)$; and any of the techniques mentioned earlier are applicable for bottomside h'-f reduction. For topside reduction the methods must be modified to include the variation of f_H with h. Doupnik (1963) and Wright (1963, personal communication) have made some progress in this area.

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[1] Lovitt, W. V. Linear Integral Equations, Dover Publications, Inc., 1950.

Now suppose $f_{min} > o$. Ther one can show that one mode alone cannot uniquely specify the electron density profile. Schmerling (personal communication) demonstrates this in the following manner: consider an h'-f record which has an $f_{min} > o$. For only the ordinary mode, one can draw a number of physically reasonable $h'_o(f)$ curves below f_{min}. If each complete $h'_o(f)$ is now reduced, a number of different $f_N(h)$ profiles will result, all of which have the same ordinary virtual heights above f_{min}. Therefore, numerically inverting equation (1.5) by using $h'_o(f)$ points above f_{min} alone cannot uniquely specify the distribution which produced them; since there are many distributions which will give them. A special case of interest is that for $f_H = o$ and a monotonic layer

$$h'_o(f) = \int_o^f \frac{f\, dh/df_N}{\sqrt{f^2 - f_N^2}}\, df_N \tag{2.1}$$

which can be inverted to give

$$h(f_N) = \frac{2}{\pi} \int_o^{f_N} \frac{h'_o(f)\, df}{\sqrt{f_N^2 - f^2}} = \frac{2}{\pi} \left\{ \int_o^{f_{min}} \frac{h'_o(f)\, df}{\sqrt{f_N^2 - f^2}} + \int_{f_{min}}^{f} \frac{h'_o(f)\, df}{\sqrt{f_N^2 - f^2}} \right\}$$

$$h_j(f_N) = \frac{2}{\pi} \left\{ \int_o^{f_{min}} \frac{h'_{o_j}(f)\, df}{\sqrt{f_N^2 - f^2}} + \int_{f_{min}}^{f} \frac{h'_o(f)\, df}{\sqrt{f_N^2 - f^2}} \right\} \tag{2.2}$$

Equation (2.2) is an analytical statement of Schmerling's proof, where the subscript j designates the different electron density profiles resulting from choosing different $h'_o(f)$ curves below f_{min} and keeping the same $h'_o(f)$ curve above f_{min}. Several workers, notably Titheridge (1959), Paul and Wright (1963), Storey (1959) and Doupnik (1963), have suggested that one can determine the profile above f_{min} by using both $h'_o(f)$ and the $h'_x(f)$ curves. The implied assumption is that there exists only one electron density profile which can produce the same $h'_o(f)$ and $h'_x(f)$ above f_{min}. We shall show that, in the case of longitidinal propagation, $\theta = 0^o$, this assumption is false, but in general their assumption may be justified.

Consider the following:

$$\theta = 0, \text{ (longitudinal propagation)}$$

$\frac{dh}{df_N}$ exists, f_H = constant, $f_R^2 = f(f + f_H)$, $\tilde{f}_R^2 = f(f - f_H)$ then

$$h'_o(f) = \int_o^{f_R} \frac{\partial}{\partial f} \left\{ f(1 - f_N^2/f_R^2)^{\frac{1}{2}} \right\} \frac{dh}{df_N} \, df_N$$

Unz (1960) has shown that $\int \frac{\partial}{\partial f} \left\{ f \mu \right\} \frac{dh}{df_N} \, df_N = \frac{\partial}{\partial f} \int f \mu \frac{dh}{df_N} \, df_N$

then $$h'_o(f) = \frac{\partial}{\partial f} \int_o^{f_R} f \left[1 - \frac{f_N^2}{f_R^2} \right]^{\frac{1}{2}} \frac{dh}{df_N} \, df_N$$

$$\int_o^f h'_o(f_1) df_1 = \int_o^f \frac{\partial}{\partial f_1} \left[f_1 \int_o^{\sqrt{f_1^2 + f_1 f_H}} \left(1 - f_N^2/(f_1^2 + f_1 f_H) \right)^{\frac{1}{2}} \frac{dh}{df_N} \, df_N \right] df_1$$

$$\int_0^f h_o'(f_1)df_1 = f\int_0^{\sqrt{f^2+ff_H}} \left(1 - f_N^2/(f^2+ff_H)\right)^{\frac{1}{2}} \frac{dh}{df_N}\, df_N = f\int_0^{f_R}\left(1-\frac{f_N^2}{f_R^2}\right)^{\frac{1}{2}}\frac{dh}{df_N}\, df_N$$

similarly

$$\int_{f_H}^f h_x'(f_1)df_1 = \int_{f_H}^f \frac{\partial}{\partial f_1}\left[f_1 \int_0^{\sqrt{f_1^2 - f_1 f_H}} \left(1 - f_N^2/(f_1^2 - f_1 f_H)\right)^{\frac{1}{2}} \frac{dh}{df_N}\, df_N\right] df_1$$

$$\int_{f_H}^f h_x'(f_1)df_1 = f\int_0^{\sqrt{f^2 - f f_H}} \left(1 - f_N^2/(f^2 - f f_H)\right)^{\frac{1}{2}} \frac{dh}{df_N}\, df_N =$$

$$= f\int_0^{\tilde{f}_R} \left(1 - \frac{f_N^2}{\tilde{f}_R^2}\right)^{\frac{1}{2}} \frac{dh}{df_N}\, df_N$$

Let us define $G(x) \equiv \int_0^x \left[1 - \frac{f_N^2}{x^2}\right]^{\frac{1}{2}} \frac{dh}{df_N}\, df_N$

then $\frac{1}{f}\int_0^f h_o'(f_1)df_1 = G\left(f_R(f)\right)$

and $\frac{1}{f}\int_{f_H}^f h_x'(f_1)df_1 = G\left(\tilde{f}_R(f)\right)$

$$G(f_R) = \frac{2}{-f_H + \sqrt{f_H^2 + 4f_R^2}} \int_{f_1=0}^{f_1 = f = \frac{-f_H + \sqrt{f_H^2+4f_R^2}}{2}} h_o'(f_1)\, df_1$$

$$\rightarrow \quad G(\tilde{f}_R) = \frac{2}{-f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}} \int_{f_1=o}^{f_1 = f = \frac{-f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}}{2}} h'_o(f_1)df_1$$

But $$G(\tilde{f}_R) = \frac{1}{f}\int_{f_H}^{f} h'_x(f_1)df_1 = \frac{2}{f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}} \int_{f_H}^{f_1 = f = \frac{f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}}{2}} h'_x(f_1)\,df_1$$

$$\therefore \frac{1}{f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}} \int_{f_1=f_H}^{f_1 = f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}} h'_x(f_1)df_1 =$$

$$= \frac{1}{-f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}} \int_{f_1=o}^{f_1 = \frac{-f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}}{2}} h'_o(f_1)\,df_1$$

if we now let $f = \frac{f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}}{2}$ then $f - f_H = \frac{-f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}}{2}$

and $2f = f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}$ also $2(f - f_H) = -f_H + \sqrt{f_H^2 + 4\tilde{f}_R^2}$

then $$\frac{1}{2f}\int_{f_H}^{f} h'_x(f_1)df_1 = \frac{1}{2(f - f_H)}\int_{o}^{(f-f_H)} h'_o(f_1)df_1$$

Taking the derivative of both sides with respect to f

$$h'_x(f) = \frac{f}{f - f_H} h'_o(f - f_H) - \frac{f_H}{(f - f_H)^2} \int_{f_1 = o}^{f_1 = f - f_H} h'_o(f_1)\, df_1$$

For $f \geq \tilde{f}_{min} - f_H = f_{min}$

$$h'_x(f) = \frac{f}{f - f_H} h'_o(f - f_H) - \frac{f_H}{(f - f_H)^2} \left\{ \int_o^{\tilde{f}_{min} - f_H} h'_o(f_1) df_1 + \right.$$

$$\left. + \int_{\tilde{f}_{min} - f_H}^{f - f_N} h'_o(f_1)\, df_1 \right\}$$

$$h'_x(f) = \frac{f}{f - f_H} h'_o(f - f_H) - \frac{f_H}{(f - f_H)^2} \left\{ A + \int_{\tilde{f}_{min} - f_H}^{f - f_H} h'_o(f_1)\, df_1 \right\}$$

where $A = \int_o^{\tilde{f}_{min} - f_H} h'_o(f_1)\, df_1$ and the ordinary curve is observed down to some f_{min} and $\tilde{f}_{min} = f_{min} + f_H$.

This equation states that one extraordinary point and the $h'_o(f)$ curve above f_{min} determine the $h'_x(f)$ curve above f_{min}. The above equation also states that at the poles ionograms having $h'_o(f)$ curves such that

$$\int_o^{\tilde{f}_{min} - f_H} h'_{o_1}(f_1) df_1 = \int_o^{\tilde{f}_{min} - f_H} h'_{o_2}(f_1) df_1 \quad \text{and} \quad h'_{o_1}(f) = h'_{o_2}(f) \text{ for } f \geq f_{min}$$

will have the same $h'_x(f)$ curves for $f \geq f_{min}$. Therefore, a number of different electron density profiles will produce the same $h'_o(f)$ curves above f_{min}. Adding the $h'_x(f)$ curve has limited the number of curves one can draw below f_{min} (to use Schmerling's argument), but it has not reduced the number to one curve. Off the poles this derivation obviously breaks down, and as Wright (personal communication) has pointed out, in practice, even close to the poles, one observes the rays reflected from X = 1 and X = 1 - Y, rather than X = 1 + Y (sometimes called the z-trace) and X = 1 -Y (sometimes called the x-trace). Under these conditions there does not seem to be a proof similar to the one presented which implies that the ordinary ray (X = 1, reflection condition) and the extraordinary ray (X = 1 - Y, reflection condition) contain exactly the same information. Wright also pointed out that model studies done by his group indicate that, at a particular latitude, μ'_o and μ'_x are approximately proportional to each other; but in general, of course, this is not true. The case at the poles seems to be similar in the sense that here again nothing new is added by considering the z-trace and x-trace together.

Since, in general, μ'_o is not proportional to μ'_x, we will assume, pending more detailed studies, that the $h'_o(f)$ and $h'_x(f)$ curves do uniquely specify the profile above f_{min}. We will therefore present a solution to the "start" problem in II-D based on an idea suggested by Titheridge (1959).

B. Minimization Techniques

The technique which we wish to apply is an iterative one where a certain quantity is minimized. Therefore, we shall first discuss minimization techniques in general.

Most of the useful methods for obtaining an approximate real solution of a real non-linear equation, of the form $f(x) = o$, involve iterative processes in which an initial approximation z_o to a desired real root $x = \alpha$ is obtained, by rough graphical methods or otherwise, and a sequence of numbers $z_o, z_1, z_2, \ldots$ is generated which converge to a limit α. The process is based upon the development of a recursion formula for z_{i+1} in terms of z_i, so that z_{i+1} may be calculated after z_i is known. The equation $y = f(x, a, b, c)$ can be cast into the above form, where we wish to determine the parameters a, b, and c such that this formula is to be a good fit to the data (x_i, y_i), $(i = 1, \ldots, m)$ and $G(a, b, c) = \sum_i^m \left| y_i - f(x_i, a, b, c) \right|^2$. If the observed experimental values exactly satisfy the assumed functional form, f, and there is no noise in the data, then there exists an a^1, b^1, and c^1 such that $G(a^1, b^1, c^1) = o$. In general, our functional form will only approximate the "real" function, and the experimental data will be noisy; we therefore seek an a, b, and c which will minimize G. Since G is usually a non-linear function of a, b, and c, iterative techniques are suitable. The three most commonly used techniques are the (1) grid search, (2) gradient, (3) method of differential correction. (3) is commonly called "least squares", which is somewhat confusing, since all three methods are based on minimizing G (the sum of the squared residuals). We have therefore adopted the nomenclature of Nielsen[1]. A brief

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[1] Nielsen, Methods in Numerical Analysis, MacMillan, 1961.

description of methods (1) and (2) is given in Kunz[1]. Method (3) will be explained in detail, since it is the technique which we applied to invert the integral equations. The following derivation was taken from Nielsen.

C. Reduction of Virtual Heights to True Heights

We here assume $f_{min} = 0$, or that the $h'(f)$ curve can be extrapolated to $f = 0$, and therefore we need consider only one mode, the ordinary.

$$h'(f) = \int_o^{h_R} \mu'(f_N^2(h),\ f_H(h),\ \theta,\ f)\ dh \qquad (2.3)$$

Here the subscript o has been suppressed. μ' is now expressed as a function of $f_N^2(h)$ for convenience. Assume now a functional form for $f_N^2(h) = f_N^2(A_1, A_2, \ldots A_n, h)$ where the A_i are undetermined parameters. One can then determine the parameters in the following manner:

At the height of reflection (for the ordinary mode)

$$f_N^2 = f^2 \rightarrow f^2 = f_N^2 (A_1, A_2, \ldots, A_n, h_R) \rightarrow F(f^2, h_R) = o$$

where $F(f^2, h_R) \equiv f^2 - f_N^2 (A_1, A_2, \ldots, A_n, h_R)$ and we assume that $F(f^2, h_R)$ defines a unique implicit function, i.e. $h_R = h_R(A_1, A_2, \ldots A_n, f^2)$.

$$h' = h'(f) = I(A_1, A_2, \ldots, A_n, f) \equiv \int_{h=o}^{h=h_R(A_1, \ldots, A_n, f^2)} \mu' \left| f, f_H(h), \theta, f_N^2(A_1, A_2, \ldots A_n, h) \right| dh$$

$$h' = I(A_1, A_2, \ldots, A_n, f) \qquad (2.4)$$

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[1] Kunz.

Here I is, in general, a non-linear function of the A_i, and equation (2.4) is to be a good fit to the data (f_j, h_j') for $j = 1, \ldots, m$. The residuals, R_j, are then given by:

$$R_1 = I(A_1, \ldots, A_n, f_1) - h_1'$$
$$R_2 = I(A_1, \ldots, A_n, f_2) - h_2'$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$R_m = I(A_1, \ldots, A_n, f_m) - h_m'$$

where h_j' are the observed experimental values. We now make an initial approximation of the parameters A_i, and call them oA_i. A number of different ways may be used to make these initial "guesses", some of which will be discussed later. We seek to determine corrections, c_i, to these approximate constants such that the corrected constants will better fit the data. Denoting the corrected constants by A_i, we have

$$_1A_i = oA_i + c_i \tag{2.5}$$

then
$$R_j = I(A_1, \ldots, A_n, f_j) - h_j' \quad j = 1, \ldots, m$$
$$R_j = I(oA_1 + c_1, \ldots, oA_n + c_n, f_j) - h_j'$$
$$R_j + h_j' = I(oA_1 + c_1, \ldots, oA_n + c_n, f_j)$$
$$\text{L.H.S.} = \text{R.H.S.}$$

Expanding the R.H.S. by Taylor's theorem for a function of several variables about the initial approximations,

$$I(oA_1 + c_1, \ldots, oA_n + c_n, f_j) = I(oA_1, \ldots, oA_n, f_j) + \sum_{i=1}^{i=n} \left(\frac{\partial I_j}{\partial A_i}\right) c_i$$

+ higher order terms in c_i

where $\left(\frac{\partial I_j}{\partial A_i}\right)_o \equiv$ the partial derivative $\frac{\partial I}{\partial A_j}$ evaluated at

$$A_i = oA_i \quad (i = 1, \ldots, m)$$

Defining $oh'_j \equiv I(oA_1, \ldots, oA_n, f_j)$ as the first approximation to the observed values, h'_j. $I_j \equiv I(oA_1, \ldots, oA_n, f_j)$

then $R_j + h'_j = oh'_j + \sum_{i=1}^{i=n} \left(\frac{\partial I_j}{\partial A_i}\right)_o c_i \qquad i = 1, \ldots, m$

where we have neglected higher order terms in c_i, which is valid if the c_i are "small".

Finally $R_j = \sum_{i=1}^{n} \left(\frac{\partial I_j}{\partial A_i}\right)_o c_i + r_j$ where $r_j = oh'_j - h'_j$

$$G(c_1, \ldots, c_n) = \sum_{j=1}^{m} R_j^2 \tag{2.6}$$

A necessary condition for G to be a minimum with respect to the c_i is that $\frac{\partial G}{\partial c_i} = o$ for every i.

$$\text{which} \rightarrow \sum_{j=1}^{m} 2R_j \frac{\partial R_j}{\partial c_i} = \sum_{j=1}^{m} 2R_j \left(\frac{\partial I_j}{\partial A_i}\right)_o = o \tag{2.7}$$

Equation (2.5) gives rise to the so called "normal" equations:

$$\sum_{j=1}^{m} R_j \left(\frac{\partial I_j}{\partial A_1}\right)_o = R_1 \left(\frac{\partial I_1}{\partial A_1}\right)_o + R_2 \left(\frac{\partial I_2}{\partial A_1}\right)_o + \ldots + R_m \left(\frac{\partial I_m}{\partial A_1}\right)_o = o$$

$$\sum_{j=1}^{m} R_j \left(\frac{\partial I_j}{\partial A_2}\right)_o = R_1 \left(\frac{\partial I_1}{\partial A_2}\right)_o + R_2 \left(\frac{\partial I_2}{\partial A_2}\right)_o + \ldots + R_m \left(\frac{\partial I_m}{\partial A_2}\right)_o = o$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{j=1}^{m} R_j \left(\frac{\partial I_j}{\partial A_n}\right)_o = R_1 \left(\frac{\partial I_1}{\partial A_n}\right)_o + R_2 \left(\frac{\partial I_2}{\partial A_n}\right)_o + \ldots + R_m \left(\frac{\partial I_m}{\partial A_n}\right)_o = o$$

$$\text{where } R_j = \sum_{i=1}^{n} \left(\frac{\partial I_j}{\partial A_i}\right)_o c_i + r_j$$

The normal equations are then:

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \left(\frac{\partial I_j}{\partial A_1}\right)_o \left(\frac{\partial I_j}{\partial A_i}\right)_o c_i + \sum_{j=1}^{m} \left(\frac{\partial I_j}{\partial A_1}\right)_o r_j = o$$

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \left(\frac{\partial I_j}{\partial A_2}\right)_o \left(\frac{\partial I_j}{\partial A_i}\right)_o c_i + \sum_{j=1}^{m} \left(\frac{\partial I_j}{\partial A_2}\right)_o r_j = o$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \left(\frac{\partial I_j}{\partial A_n}\right)_o \left(\frac{\partial I_j}{\partial A_i}\right)_o c_i + \sum_{j=1}^{m} \left(\frac{\partial I_j}{\partial A_n}\right)_o r_j = o$$

We now have n linear equations with n unknowns, c_i. Having determined the c_i from the normal equations, we can then determine the ${}_1A_i$ from equation (2.5). The ${}_1A_j$ can then be used as approximate values, and the whole procedure repeated until ${}_{n-1}A_i/{}_nA_i$ is less than some predetermined delta (convergence criterion The ${}_nA_i$ are the "best" layer constants for the assumed functional form in the least squares sense. Inspecting the normal equations, we see that the coefficients of the c_i are sums over the partial derivatives of $I(A_1, \ldots, A_n, f)$ with respect to the A_i. We shall now derive the expressions for these partial derivatives.

To simplify matters we shall consider a one parameter model, i.e. $f_N(h) = g(A, h)$.

$$\text{Then} \quad I(f, A) = \int_o^{h_R(A,f)} \mu'(f_H, f, g(A,h), \theta)\, dh \quad . \tag{2.8}$$

Also let $g(A, h)$ be a monotonic function of h (this restriction will be removed later). Then

$$I(f, A) = \int_o^f \mu'(f_H, f, f_N, \theta) \frac{dh}{df_N} \, df_N$$

$$\text{where } \frac{dh}{df_N} = S(A, f_N) \text{ and } \frac{df_H}{dh} = o$$

$$\frac{\partial I}{\partial A} = \frac{\partial}{\partial A} \int_o^f \mu'(f_H, f, f_N, \theta) \frac{dh}{df_N} \, df_N; \text{ let } f_N = f \sin \phi$$

$$\frac{\partial I}{\partial A} = \frac{\partial}{\partial A} \int_o^{\pi/2} f \mu'(f_H, f, \sin \phi, \theta) \cos \phi \frac{dh}{df_N} (\phi)\, d\phi$$

$$\frac{dh}{df_N} (\phi) = S(A, f \sin \phi)$$

It can be shown that $\lim_{\phi \to \pi/2} \mu' \cos \phi = \frac{1}{\sin \theta}$, where θ is the geomagnetic dip angle (Schmerling (1957) equation 4.20). This limit exists for every $\theta \neq o$. The function $H(f, \phi) \equiv \mu' \cos \phi$ is undefined at the upper limit, $\pi/2$. We therefore define it to be $\frac{1}{\sin\theta}$, i.e.

$$H(f, \phi) = \begin{Bmatrix} \frac{1}{\sin\theta} \; ; \; \phi = \pi/2 \\ \\ \mu' \cos\phi \; ; \; o \leq \phi < \pi/2 \end{Bmatrix}$$

$$\frac{\partial I}{\partial A} = \frac{\partial}{\partial A} \int_o^{\pi/2} fH(f, \phi) \frac{dh}{df_N} (\phi)\, d\phi$$

If $\frac{\partial^2 h}{\partial A \partial f_N}$ exists and is continuous, and $\frac{dh}{df_N}$ is continuous, then $fH(f, \phi) \frac{dh}{df_N} (\phi)$ and $\frac{\partial}{\partial A} \left\{ fH(f, \phi) \frac{dh}{df_N} (\phi) \right\}$ are continuous in the rectangle R: $\begin{matrix} \alpha \leq A \leq \beta \\ \pi/2 \leq \phi \leq o \end{matrix}$. These conditions are sufficient to differentiate under the integral sign[1].

$$\frac{\partial I}{\partial A} = \frac{\partial}{\partial A} \int_o^{\pi/2} fH(f, \phi) \frac{dh}{df_N} (\phi)\, d\phi = \int_o^{\pi/2} fH(f, \phi) \frac{\partial}{\partial A} \left(\frac{dh}{df_N} (\phi) \right) d\phi$$

$$\frac{\partial I}{\partial A} = \int_o^{\pi/2} f\, \mu'(f_H, f, \sin\phi, \theta) \cos\phi \frac{\partial^2 h}{\partial A \partial f_N}\, d\phi \; . \qquad (2.9)$$

In general, it is convenient to re-formulate equation (2.9) in terms of derivatives of f_N^2 rather than derivatives of h. We shall now derive such an expression.

$$\frac{\partial I}{\partial A} = \int_o^{f} \mu' (f_H, f, f_N) \frac{\partial^2 h}{\partial A \partial f_N}\, df_N \; . \qquad (2.10)$$

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[1] Brand, <u>Advanced Calculus</u>, Wiley, 1955.

Assuming that $\frac{\partial^2 h}{\partial A \, \partial f_N} = \frac{\partial^2 h}{\partial f_N \partial A}$

$$\frac{\partial I}{\partial A} = \int_o^f \mu' \, (f_H, f, f_N) \frac{\partial^2 h}{\partial f_N \partial A} \, df_N = \int_o^f \mu' \, (f_H, f, f_N) \frac{\partial}{\partial f_N} \left|\frac{\partial h}{\partial A}\right| \, df_N$$

$$\frac{\partial I}{\partial A} = \int_o^{h_R} \mu' \, (f_H, f, f_N) \frac{\partial}{\partial f_N} \left|\frac{\partial h}{\partial A}\right| \frac{df_N}{dh} \, dh = \int_o^{h_R} \mu' \, (f_H, f, f_N) \frac{\partial}{\partial h} \left|\frac{\partial h}{\partial A}\right| dh$$

$$\frac{\partial I}{\partial A} = \int_o^{h_R} \mu' \, \left|f_H, f, f_N(A, h)\right| \frac{\partial}{\partial h} \left\{ \frac{\left(\frac{\partial f_N}{\partial A}\right)}{\left(\frac{\partial f_N}{\partial h}\right)} \right\} dh. \tag{2.11}$$

Let $f_N(A, h) = g(A, h)$. Equation (2. 11) is valid for monotonic layers and a constant magnetic field as a function of h.

We will now remove the monotonic assumption. Suppose $f_N(A, h)$ is a non-monotonic function, then $g(A, h)$ is a non-monotonic function of h. We will only consider those h'(f) whose probe frequencies, f, have h_R's which do not satisfy the equation $\frac{dg}{dh} = o$, i. e. we will not consider those frequencies whose heights of reflection are extrema of the model. For physically realistic models, this condition throws out a negligible number of virtual heights.

Then $\quad I(A,f) = \int_o^{h_R} \mu' \, dh = \int_o^{h_1} \mu' \, dh + \int_{h_1}^{h_R} \mu' \, dh$

where $g(A, h)$ is a monotonic function of h for $h_1 \leq h \leq h_R$.

Then $\frac{\partial I}{\partial A} = \frac{\partial}{\partial A} \int_o^{h_1} \mu \quad dh + \frac{\partial}{\partial A} \int_{h_1}^{h_R} \mu' \, dh$

where μ' and $\frac{\partial \mu'}{\partial f_N}$ are continuous in the rectangle R: $o \leq h \leq h_1$, $\alpha \leq A \leq \beta$.

$$\frac{\partial I}{\partial A} = \int_o^{h_1} \frac{\partial \mu'}{\partial f_N} \frac{\partial f_N}{\partial A} + \mu' \Big|_{h=h_1} \frac{\partial h_1}{\partial A} - \int_{h_1}^{h_R} \mu' \frac{\partial}{\partial h} \left\{ \frac{(\partial f_N / \partial A)}{(\partial f_N / \partial h)} \right\} dh . \quad (2.12)$$

where h_1 is any height such that g(A, h) is monotonic for $h \geq h_1$. We have shown that the derivatives of I with respect to the layer constants, A_i, exist for both monotonic and non-monotonic functions of h, and derived the explicit formulae for these derivatives, for a constant magnetic field, i.e. these formulae are applicable to bottomside ionogram reduction.

For topside analysis, we cannot ignore the variation of gyro-frequency with height, and shall therefore derive the equations for the derivatives of I with respect to the layer constants for a monotonic layer (ionograms from the topside satellite Alouette indicate that this is generally the case) and a variable gyro-frequency.

$$I(A, f) = \int_{h=h_o(A)}^{h=h_R(A, f)} \mu' \left(f, f_H(h), \theta, g(A, h) \right) dh \qquad (2.13)$$

where equation (2.12) is a convergent improper integral and

$$I(A, f) = \lim_{\epsilon \to o} \int_{h_o}^{h_R - \epsilon} \mu' \, dh$$

$$\frac{\partial I}{\partial A} = \frac{\partial}{\partial A} \lim_{\epsilon \to o} \int_{h_o}^{h_R - \epsilon} \mu' \, dh .$$

In order to interchange the derivative operator and the limit operator, one must show that 2.12 converges, and that $\int_{h_o}^{h_R} \frac{\partial \mu'}{\partial A}\, dh$ converges uniformly[1], which for the complicated integrand, μ', is rather difficult. We shall assume that the derivation which led to equation (2.12) serves as a plausibility proof that this interchange is permissible.

Then $\frac{\partial I}{\partial A} = \lim_{\varepsilon \to o} \frac{\partial}{\partial A} \int_{h_o}^{h_R-\varepsilon} \mu'\, dh$

$$\frac{\partial I}{\partial A} = \lim_{\varepsilon \to o} \int_{h_o}^{h_R-\varepsilon} \left(\frac{\partial \mu'}{\partial f_N^2}\right)\left(\frac{\partial f_N^2}{\partial A}\right) dh + \Big[\mu'\Big]_{h=h_R-\varepsilon} \frac{\partial (h_R-\varepsilon)}{\partial A} - \Big[\mu'\Big]_{h=h_o} \frac{\partial h_o}{\partial A}$$

Since μ' and $\frac{\partial \mu'}{\partial A}$ are continuous in the rectangle R: $o \leq h \leq h_R - \varepsilon$, $\alpha \leq A \leq \beta$.

Consider: $\frac{\partial}{\partial h}\left\{\mu'\left[\left(\frac{\partial f_N^2}{\partial A}\right)\Big/\left(\frac{\partial f_N^2}{\partial h}\right)\right]\right\}$ where for a monotonic layer $\left(\frac{\partial f_N^2}{\partial h}\right) \neq o$ except at $h=h_{max}$.

$$= \left(\frac{\partial \mu'}{\partial h}\right)\left[\left(\frac{\partial f_N^2}{\partial A}\right)\Big/\left(\frac{\partial f_N^2}{\partial h}\right)\right]\Big\} + \mu' \frac{\partial}{\partial h}\left[\left(\frac{\partial f_N^2}{\partial A}\right)\Big/\left(\frac{\partial f_N^2}{\partial h}\right)\right]$$

where $\left(\frac{\partial \mu'}{\partial h}\right) = \left(\frac{\partial \mu'}{\partial f_N^2}\right)\left(\frac{\partial f_N^2}{\partial h}\right) + \left(\frac{\partial \mu'}{\partial f_H}\right)\left(\frac{\partial f_H}{\partial h}\right)$

and $\left(\frac{\partial \mu'}{\partial f_N^2}\right)\left(\frac{\partial f_N^2}{\partial h}\right)\left[\left(\frac{\partial f_N^2}{\partial A}\right)\Big/\left(\frac{\partial f_N^2}{\partial h}\right)\right] = \left(\frac{\partial \mu'}{\partial f_N^2}\right)\left(\frac{\partial f_N^2}{\partial A}\right)$

$$\therefore \frac{\partial}{\partial h}\left\{\mu'\left[\left(\frac{\partial f_N^2}{\partial A}\right)\Big/\left(\frac{\partial f_N^2}{\partial h}\right)\right]\right\} = \left(\frac{\partial \mu'}{\partial f_N^2}\right)\left(\frac{\partial f_N^2}{\partial A}\right) +$$

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[1] Brand, <u>Advanced Calculus</u>, Wiley, 1955.

$$+ \left(\frac{\partial \mu'}{\partial f_H}\right) \left(\frac{\partial f_H}{\partial h}\right) \left[\left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right] + \mu' \frac{\partial}{\partial h} \left[\left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right]$$

where $\left(\frac{\partial \mu'}{\partial f_N^2}\right) \left(\frac{\partial f_N^2}{\partial A}\right) = \frac{\partial \mu'}{\partial A}$

$$\therefore \frac{\partial \mu'}{\partial A} = \frac{\partial}{\partial h} \left\{ \mu' \left[\left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right] \right\} - \left(\frac{\partial \mu'}{\partial f_H}\right) \left(\frac{\partial f_H}{\partial h}\right) \left[\left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right]$$

$$- \mu' \frac{\partial}{\partial h} \left[\left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right]$$

$$\therefore \int_{h=h_o}^{h=h_R-\epsilon} \frac{\partial \mu'}{\partial A}\, dh = \int_{h=h_o}^{h=h_R-\epsilon} \frac{\partial}{\partial h} \left\{ \mu' \left[\left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right] \right\} dh$$

$$- \int_{h=h_o}^{h=h_R-\epsilon} \left(\frac{\partial \mu'}{\partial f_H}\right) \left(\frac{\partial f_H}{\partial h}\right) \left[\left(\frac{\partial f_N^2}{\partial h}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right] dh$$

$$- \int_{h=h_o}^{h=h_R-\epsilon} \mu' \frac{\partial}{\partial h} \left\{ \left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right\} dh$$

where $-\left(\frac{\partial h}{\partial A}\right) = \left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right)$

$$\therefore \int_{h=h_o}^{h=h_R-\epsilon} \frac{\partial}{\partial h} \left\{ \mu' \left[\left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right] \right\} dh = -\left[\mu' \frac{\partial h}{\partial A} \right]_{h=h_R-\epsilon} + \left[\mu' \frac{\partial h}{\partial A} \right]_{h=h_o}$$

and, for the ordinary ray $\left[\frac{\partial h}{\partial A} \right]_{h=h_R} = \frac{\partial h_R}{\partial A}$

since $h = h(f_N^2, A)$ and $h_R \equiv h(f^2, A)$ for the ordinary mode.

$$\therefore \quad \frac{\partial h}{\partial A} = \frac{\partial h(f_N^2, A)}{\partial A} \rightarrow \left[\frac{\partial h}{\partial A}\right]_{h=h_R} = \left[\frac{\partial h(f_N^2, A)}{\partial A}\right]_{f_N^2=f^2} = \frac{\partial h(f^2, A)}{\partial A}$$

but $\quad h_R \equiv h(f^2, A) \quad \therefore \quad \dfrac{\partial h(f^2, A)}{\partial A} = \dfrac{\partial h_R}{\partial A}$

$$\rightarrow \left[\frac{\partial h}{\partial A}\right]_{h=h_R} = \frac{\partial h_R}{\partial A} \rightarrow \left[\frac{\partial h}{\partial A}\right]_{h=h_R-\epsilon} = \frac{\partial (h_R-\epsilon)}{\partial A}$$

$$\left(\frac{\partial I}{\partial A}\right)_f = \underset{\epsilon \rightarrow o}{\text{limit}} \int_{h_o}^{h_R-\epsilon} \frac{\partial \mu'}{\partial A}\, dh + \left[\mu'\right]_{h=h_R-\epsilon} \frac{\partial (h_R-\epsilon)}{\partial A} - \left[\mu'\right] \frac{\partial h_o}{\partial A}$$

where we have shown that

$$\int_{h=h_o}^{h=h_R-\epsilon} \frac{\partial \mu'}{\partial A}\, dh = -\left[\mu' \frac{\partial h}{\partial A}\right]_{h=h_R-\epsilon} + \left[\mu' \frac{\partial h}{\partial A}\right]_{h=h_o}$$

$$- \int_{h=h_o}^{h=h_R-\epsilon} \left(\frac{\partial \mu'}{\partial f_H}\right) \left(\frac{\partial f_H}{\partial h}\right) \left[\left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right] dh$$

$$- \int_{h=h_o}^{h=h_R-\epsilon} \mu' \frac{\partial}{\partial h} \left\{ \left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right\} dh$$

$$\therefore \left(\frac{\partial I}{\partial A}\right)_f = \underset{\epsilon \rightarrow o}{\text{limit}} \; - \int_{h=h_o}^{h=h_R-\epsilon} \mu' \frac{\partial}{\partial h} \left\{ \left(\frac{\partial f_N^2}{\partial A}\right) \Big/ \left(\frac{\partial f_N^2}{\partial h}\right) \right\} dh$$

$$- \int_{h=h_o}^{h=h_R-\epsilon} \left(\frac{\partial \mu'}{\partial f_H}\right) \left(\frac{df_H}{dh}\right) \left\{ \left(\frac{\partial f_N^2}{\partial A}\right) \bigg/ \left(\frac{\partial f_N^2}{\partial h}\right) \right\} dh$$

$$\frac{\partial I}{\partial A} = - \int_{h=h_o}^{h=h_R} \mu' \frac{\partial}{\partial h} \left\{ \left(\frac{\partial f_N^2}{\partial A}\right) \bigg/ \left(\frac{\partial f_N^2}{\partial h}\right) \right\} dh - \int_{h=h_o}^{h=h_R} \left(\frac{\partial \mu'}{\partial f_H}\right) \left(\frac{df_H}{dh}\right) \left\{ \left(\frac{\partial f_N^2}{\partial A}\right) \bigg/ \left(\frac{\partial f_N^2}{\partial h}\right) \right\} dh \qquad (2.13a)$$

Equation (2.13a) is valid for a monotonic layer with variable gyrofrequency for the ordinary mode, and can obviously be extended to non-monotonic layers in the manner of equation (2.12).

We have derived the system of equations necessary to reduce top and bottomside ionograms for $f_{min} = 0$. We shall now consider the "start problem".

D. <u>"Start" Problem</u>

$$h_o'(f) = \int_0^{h_{R_o}} \mu_o' \, dh = \int_0^{h_{min_o}} \mu_o' \, dh + \int_{h_{min_o}}^{h_{R_o}} \mu_o' \, dh \qquad (2.14)$$

$$h_x'(f) = \int_0^{h_{R_x}} \mu_x' \, dh = \int_0^{h_{min_x}} \mu_x' \, dh + \int_{h_{min_x}}^{h_{R_x}} \mu_x' \, dh \, . \qquad (2.15)$$

where h_{min_o} is the height of reflection of the ordinary ray at f_{min_o}, and h_{min_x} is the height of reflection of the extraordinary ray at f_{min_x}. We shall assume that $h_{min_o} = h_{min_x} = h_{min}$, i.e. $f^2_{min_o} = f_{min_x}$ $(f_{min_x} - f_H)$.

There are two approaches to the start problem:

(1) Paul and Wright (1963) and Doupnik (1963) assume that the layer is monotonic for $o \leq h \leq h_{min}$ and therefore derive from equation (2.14) and equation (2.15) the following

$$h'_o(f) = \int_o^{f_R} \mu'_o \frac{dh}{df_N} \, df_N = \int_o^{f_{min}} \mu'_o \frac{dh}{df_N} \, df_N + \int_{f_{min}}^{f_R} \mu'_o \frac{dh}{df_N} \, df_N \, . \tag{2.16}$$

for $f \geq f_{min}$

$$h'_x(f) = \int_o^{\tilde{f}_R} \mu'_x \frac{dh}{df_N} \, df_N = \int_o^{f_{min}} \mu'_x \frac{dh}{df_N} + \int_{f_{min}}^{\tilde{f}_R} \mu'_x \frac{dh}{df_N} \, df_N \, . \tag{2.17}$$

They then assume a model form for $\frac{dh}{df_N}$ for $o \leq f_N \leq f_{min}$ and determine the constants from a system of linear equations using both ordinary and extraordinary virtual heights. Once this is done, their standard techniques are used to determine the rest of the electron density distribution.

(2) Titheridge (1959) suggested that one approximate μ' for $o \leq h \leq h_{min}$ by a low order polynomial in X, and allow for the underlying ionization below f_{min} by computing the moments of the distribution,

$\int_o^{h_{min}} N^k(h) \, dh$, from the ordinary and extraordinary virtual heights.

We shall extend this technique using a better approximation which provides an improved representation of the underlying ionization.

$$h'(f) = \int_o^{h_R} \mu'\left(f, f_H, f_N(h)\right) dh \rightarrow \int_o^{h_{min}} \mu'\left(x(h), y\right) dh + \int_{h_{min}}^{h_R} \mu'\left(f, f_H, f_N(h)\right) dh$$

where $x(h) = f_N^2(h)/f^2$ and $y = f_H/f$

assuming that, for the bottomside, $df_H/dh = o$

$$\text{Let } \mu' = \sum_j^m A_j(y)\, x^j \quad \text{for} \quad o \leq h \leq h_{min}$$

and $f_N^2(h) = g(A_1, \ldots, A_n, h)$ for $h_{min} \leq h \leq h_R$. (We need only consider the ordinary ray since similar results can be derived for the extraordinary.) We wish an accurate approximation for μ', having a small number of terms over the interval $o \leq x \leq 0.8$. Since this approximation will be integrated over the interval $o \leq h \leq h_{min}$, the approximation must be uniformly good over the interval. Obviously, this excludes Taylor expansions which are essentially point expansions, and suggests an orthogonal expansion. Lanczos[1] recommends the Chebyshev polynomials as the orthogonal polynomials having the maximum rate of convergence, i.e. they approximate the function over an interval, within some delta, with fewer terms than any other orthogonal expansion. Consider the case $f_H = 0.0$. Then $\mu' = \frac{1}{\sqrt{1 - X}}$, and we wish to obtain an expansion of the form $\mu' = \sum_{n=o}^{m} a_n T_n(X)$

[1] Lanczos, Applied Analysis, Prentice Hall, 1961.

where $a_o = \frac{1}{\pi} \int_{-1}^{1} \frac{\mu'}{\sqrt{1 - X^2}} dX$; $a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{\mu' T_n(X)}{\sqrt{1 - X^2}} dX \quad n \geq 1$

and $T_n(X)$ are the Chebyshev polynomials over the interval $-1 \leqslant X \leqslant 1$, commonly called the Chebyshev polynomials of the first kind.

$\mu' = \frac{1}{\sqrt{1 - X}}$ is defined for $o \leq X < 1$. Normalizing the interval of approximation, $o \leq X \leq 0.8$, let $X = 0.8 X_1$ for $o \leq X_1 \leq 1$ then

$\mu' = \frac{1}{\sqrt{1 - 0.8 X_1}}$ where $o \leq X_1 \leq 1$. Now transform to the interval $[-1, 1]$ by letting $X_2 = 2X_1 - 1$ for $o \leq X_1 \leq 1$. Then $-1 \leq X_2 \leq 1$ and

$\mu' = \frac{1}{\sqrt{1 - 0.4(X_2 + 1)}}$ where $-1 \leq X_2 \leq 1$.

Once the Chebyshev expansion has been obtained it can be rewritten as a polynomial in X. We then have the following:

$$h'(f) = \int_{o}^{h_{min}} \sum_{h=o}^{m} b_n X^n \, dh + \int_{h_{min}}^{h_R} \mu' \, dh \quad . \tag{2.18}$$

Where the b_n are functions of $y = f_H/f$ in general. Equation (2.18) holds for all frequencies, f, such that $f \geq \frac{f_{min}}{\sqrt{0.8}}$

$$h' (f) = \int_{o}^{h_{min}} \left(\sum_{n=o}^{m} \frac{b_n f_N^{2\,n}}{f^{2\,n}} \right) dh + \int_{h_{min}}^{h_R} \mu' dh = \sum_{n=o}^{m} \frac{b_n}{f^{2\,n}} \int_{o}^{h_{min}} f_N^{2\,n}(h) \, dh +$$

$$+ \int_{h_{min}}^{h_R} \mu' \, dh$$

Let $\int_{o}^{h_{min}} f_N^{2n}(h)\, dh = c_n$ and $\frac{b_n}{f^{2n}} = D_n(f)$

then
$$h'(f) = \sum_{n=o}^{m} D_n(f)\, c_n + \int_{h_{min}}^{h_R} \mu' \, dh \; . \qquad (2.19)$$

Where the $D_n(f)$ are known functions of f. We now choose a model for $f_N^2(h)$ for $h_{min} \leq h \leq h_{max}$ (h_{max} is the height of the maximum electron density of the layer). One now must determine the model constants in addition to the c_n. A similar equation can be derived for the extraordinary virtual heights. One then may use these two equations to determine the unknown constants in the manner described in II-C. Also, if the ionogram indicates that $f_N(h)$ is a monotonic function of h for $h \geq h_{min}$, one may rewrite equation (2.19) in the following manner:

$$h'(f) = \sum_{n=o}^{m} D_n(f)\, c_n + \int_{f_{min}}^{f_R} \mu' \frac{dh}{df_N}\, df_N \; . \qquad (2.19)$$

The methods of Paul and Wright, Titheridge, or Doupnik may now be used. Since one can only determine a few terms of $\sum_{n=o}^{m} D_n(f) c_n$, i.e. m typically 3 or 4, from the ordinary and extraordinary virtual heights, it is imperative that the expansion of μ' be a quickly convergent one, i.e. a Chebyshev expansion. It is obvious from the above discussion that no model assumption has been made in the unseen range ($f \leq f_{min}$) and therefore this approach is inherently superior to the existing "start" solutions which do make these assumptions. One need only examine the results of the model studies of Doupnik (1963) to realize the sensitivity of model approaches below f_{min} to the particular model choosen. The above technique will work for both monotonic and non-monotonic profiles below f_{min}.

Not only do the c_n enable us to compensate for low lying ionization, but they also determine uniquely the monotonic profile which produces the correct retardation up to h_{min}. Consequently, if the profile is monotonic below f_{min}, we can determine this profile from the c_n in the following manner: for a monotonic profile, $\frac{dh}{df_N^2}$ exists. Now let

$$\zeta \equiv \frac{2f_N^2}{f_{min}^2} - 1 \quad \rightarrow \quad \text{for } o \leq f_N^2 \leq f_{min}^2 \text{ then } -1 \leq \zeta \leq 1$$

$$\therefore\ d\zeta = \frac{2\, df_N^2}{f_{min}^2} \quad \rightarrow \quad \frac{dh}{df_N^2} = \frac{2}{f_{min}^2} \frac{dh}{d\zeta} \quad \rightarrow \quad \frac{dh}{d\zeta} = \frac{f_{min}^2}{2} \frac{dh}{df_N^2}$$

$$\frac{dh}{d\zeta} = \sum_{r=o}^{m} a_r P_r(\zeta) \quad . \tag{2.20}$$

where the P_r are the Legendre polynomials, and

$$a_r = \frac{2r+1}{2} \int_{-1}^{1} \frac{dh}{d\zeta} P_r(\zeta)\, d\zeta = \frac{2r+1}{2} \int_{-1}^{1} \frac{dh}{d\zeta} \left(\sum_{j=o}^{r} b_j \zeta^j \right) d\zeta$$

$$a_r = \frac{2r+1}{2} \sum_{j=o}^{r} b_j \int_{-1}^{1} \frac{dh}{d\zeta} \zeta^j\, d\zeta$$

$$\int_{-1}^{1} \frac{dh}{d\zeta} \zeta^j \, d\zeta = \int_{o}^{f^2_{min}} \frac{dh}{df^2_N} \left(\frac{2 f^2_N}{f^2_{min}} - 1 \right)^j df^2_N$$

$$a_r = \frac{2r+1}{2} \sum_{j=o}^{r} b_j \int_{o}^{f^2_{min}} \frac{dh}{df^2_N} \left(\frac{2f^2_N}{f^2_{min}} - 1 \right)^j df^2_N$$

where $\displaystyle\int_{o}^{f^2_{min}} \frac{dh}{df^2_N} \left(\frac{2f^2_N}{f^2_{min}} - 1 \right)^j df^2_N = \sum_{\ell=o}^{j} d_\ell \left(\frac{2}{f^2_{min}} \right)^\ell \int_{o}^{f^2_{min}} \frac{dh}{df^2_N} f_N^{2\ell} df^2_N$

$$a_r = \frac{2r+1}{2} \sum_{j=o}^{r} \sum_{\ell=o}^{j} b_j d_\ell S^\ell \text{ where } S^\ell \equiv \left(\frac{2}{f^2_{min}} \right) \int_{o}^{f^2_{min}} \frac{dh}{df^2_N} f_N^{2\ell} df^2_N$$

But $\displaystyle c_n = \int_{o}^{h_{min}} f_N^{2n} (h) dh = \int_{o}^{f^2_{min}} f_N^{2n} \frac{dh}{df^2_N} df^2_N$ for a monotonic layer

$$\rightarrow S^\ell = \left(\frac{2}{f^2_{min}} \right)^\ell c_\ell$$

$$a_r = \frac{2r+1}{2} \sum_{j=o}^{r} \sum_{\ell=o}^{j} b_j \, d_\ell \frac{2^\ell c_\ell}{f^{2\ell}_{min}} . \tag{2.21}$$

$$\frac{dh}{df^2_N} = \frac{2}{f^2_{min}} \frac{dh}{d\zeta} = \frac{2}{f^2_{min}} \sum_{r=o}^{m} a_r P_r (\zeta)$$

$$\frac{dh}{df^2_N} = \sum_{r=o}^{m} \sum_{j=o}^{r} \sum_{\ell=o}^{j} (2r+1) \, b_j \, d_\ell \frac{2^\ell c_\ell}{f^{2\ell+1}_{min}} . \tag{2.22}$$

Where $\displaystyle c_\ell = \int_{o}^{h_{min}} f_N^{2\ell} (h) \, dh$ for a monotonic layer. Given the

c_ℓ we can expand $\frac{dh}{df_N^2}$ in a Legendre expansion for $o \leq f_N^2 \leq f_{min}$

and $h(f_N^2) = \int_{f^2_{min}}^{f^2_N} \frac{dh}{dR}\, dR$ where $h(f^2_{min}) = c_o$.

E. "Valley" Problem

The techniques which we have developed thus far allow us to approach the valley problem in one of two ways. One way is that suggested by Wright (private communication), which is to treat the valley problem as a "low frequency cut off" problem where f_{min} is now the critical frequency of the lowest maximum of ion density. The other approach is to assume a non-monotonic function of h and attempt to determine the constants in the manner prescribed in part C. Which is the more feasible can only be determined by model studies.

III. MODEL STUDIES

A. Purpose of Model Studies

Model studies were conducted to test the method developed in II-C. Ordinary virtual heights were computed from a parabolic layer, i.e.

$$f_N^2 = f_P^2 \left\{ 1 - \frac{(h-h_m)^2}{a^2} \right\} . \tag{3.1}$$

$f_P \equiv$ penetration frequency of the layer

$h_m \equiv$ height of maximum electron density

$a \equiv$ semi-thickness of the layer.

We then attempted to derive the layer constants for this model from the computed virtual heights. Having chosen the functional form, i.e. a parabola in h, the problem is how to make initial estimates of the layer parameters. f_P, of course, can be read off the ionogram. The simplest way to determine h_m and a (probably the worst) is to guess them. This was done because we wished to determine the effects of the initial values of the layer constants on the final values. As with most minimization routines, this routine will be trapped by the first relative minimum it finds. Therefore, one or more re-runs starting with different initial values of the parameters is recommended as a standard procedure where possible. This procedure will find a new relative minimum, or confirm the accuracy to which the original one has been located. It should be noted at this point that the technique developed in II-C is one for determining a model for the whole layer. Schmerling has pointed out that, in general, one will not be able to assume a simple functional form which will fit the whole layer. But the extension of II-C to a lamination procedure as used by Paul and Wright and Doupnik is obvious and hence will not be given. For this test, four sets of "guessed" layer constants were used with the same model. In addition, to determine the effect of "noise" on the method, three sets of input data were used: exact virtual heights, virtual heights rounded to one km, and virtual heights rounded to five km. One would also like to know the minimum number of virtual heights which will give accurate results, and, the number of input virtual heights were, therefore, varied from six to twelve. Only ordinary virtual heights were used.

B. Numerical Techniques

One must numerically evaluate integrals of the form $\int_0^{h_R} \mu' \, dh$ and $\int_0^{h_R} \mu' \, g(h) dh$; since we do not wish to restrict ourselves to monotonic models, we cannot make the substitution $dh = \frac{dh}{df_N^2} \, df_N^2$.

Initially Gaussian Quadrature was used to perform the integration, but the integrand goes to infinity at the upper limit; therefore, the standard Gaussian Quadrature formulae are inadequate (inadequate in the sense that they require large amounts of computer time).

Budden[1] has shown that

$$h'(f) = \int_0^{h_R} \mu \, dh + f \frac{\partial}{\partial f} \int_0^{h_R} \mu \, dh \; .$$

[1] Budden (op. cit).

where μ is finite and continuous over the entire range of integration $\int_o^{h_R} \mu \, dh$ was integrated using an extended five point Gaussian Quadrature method with ten intervals.

To evaluate $f \frac{\partial}{\partial f} \int_o^{h_R} \mu \, dh$, we used the finite difference formula.

$$f \frac{\partial}{\partial f} \int_o^{h_R} \mu \, dh \doteq f \frac{\Delta \int_o^{h_R} \mu \, dh}{\Delta f} = \frac{f \left(\int_o^{h_R(f_2)} \mu(f_2) dh - \int_o^{h_R(f_1)} \mu(f_1) dh \right)}{f_2 - f_1}$$

where $f_2 = f + 0.001$; $f_1 = f - 0.001$. Comparing the virtual heights given by this method with those obtained analytically (a parabolic layer in h with no magnetic field), the error was on the average about 10^{-3} km over the complete range of $o \leq f \leq f_P$.

The program which we used is "A Generalized Least Squares Program for the I. B. M. 7090 Computer" by M. H. Lietzke, as modified by Yoder of this University for the I. B. M. 7074. This program requires only the function and the partial derivatives of the function with respect to the coefficients. In our case the function is

$$I(f_P, h_m, a, f) = h_m - a + \int_{h_m - a}^{h_R(f_P, h_m, a, f)} \mu'\left[f, f_H, f_N^2, (h), \theta\right] dh; \frac{df_H}{dh} = o \tag{3.2}$$

$$\text{where } f_N^2(h) = f_P^2 \left\{ 1 - \frac{(h-h_m)^2}{a^2} \right\}$$

and $h_R = h_m - a \, (1 - f^2/f_P^2)^{\frac{1}{2}}$; let $h_o = h_m - a$

$$\frac{\partial I}{\partial h_m} = 1.0 + \int_{h_o}^{h_R} \mu' \frac{\partial}{\partial h} \left\{ - \frac{\left(\frac{\partial f_N^2}{\partial h_m}\right)}{\left(\frac{\partial f_N^2}{\partial h}\right)} \right\} dh = 1.0$$

$$\frac{\partial I}{\partial a} = -1.0 + \int_{h_o}^{h_R} \mu' \frac{\partial}{\partial h} \left\{ - \frac{\left(\frac{\partial f_N^2}{\partial a}\right)}{\left(\frac{\partial f_N^2}{\partial h}\right)} \right\} dh = -1.0 + \frac{1}{a} \int_{h_o}^{h_R} \mu' \, dh$$

$$\frac{\partial I}{\partial f_P} = \int_{h_o}^{h_R} \mu' \frac{\partial}{\partial h} \left\{ - \frac{\left(\frac{\partial f_N^2}{\partial f_P}\right)}{\left(\frac{\partial f_N^2}{\partial h}\right)} \right\} dh = \frac{a^2}{f_P} \int_{h_o}^{h_R} \mu' \left\{ \frac{1}{h-h_m} - \frac{h-h_m}{a^2} \right\} dh$$

C. Results and Discussion of the Model Studies

The sets of initial guesses used were: ($f_P = 6.0$, $h_m = 156$, $a = 134$); ($f_P = 6.0$, $h_m = 453$, $a = 38$); ($f_P = 6.0$, $h_m = 217$, $a = 22$); ($f_P = 6.50$, $h_m = 350$, $a = 100$) . The actual values of the model constants were $f_P = 6.0$, $h_m = 300.0$, $a = 50.0$, in all cases, $f_H = 1.682$ and $\theta = 0.52360$.

All sets of data converged to the actual layer constants within five iterations (see tables, where Res = difference between calculated and observed virtual heights). The results indicate the following:

(a) Convergence was unaffected by "guessing" values of the layer constants.

(b) The minimum number of points which could be used with consistent results was six.

Number of Iterations	Parameter Values f_P	h_m	a	$\sum_{i=1}^{12} (\text{Res.})_i^2$
0	6.0000	217.00	32.000	0.62243×10^5
1	6.0168	300.30	50.600	0.51117×10^1
2	6.0067	299.64	50.606	0.56060×10^1
3	6.0072	300.27	50.606	0.51780×10^1

Table 1. Determination of parabolic layer constants, using 12 data points rounded to 1 km by the method of least squares.

Number of Iterations	Parameter Values f_P	h_m	a	$\sum_{i=1}^{12} (\text{Res.})_i^2$
0	6.5000	350.00	100.00	0.36079×10^4
1	5.8269	284.86	34.822	0.44207×10^3
2	5.9354	297.42	47.540	0.43587×10^2
3	5.9964	299.09	50.229	0.15599×10^1
4	6.0074	299.50	50.622	0.92427×10^0
5	6.0073	299.50	50.627	0.92499×10^0

Table 2. Determination of parabolic layer constants, using 12 data points rounded to 1 km, by the method of least squares.

Number of Iterations	Parameter Values f_P	h_m	a	$\sum_{i=1}^{6} (\text{Res.})_i^2$
0	6.000	453.00	38.000	0.15033×10^6
1	5.999	299.09	49.872	0.48342×10^1
2	5.999	298.88	49.870	0.48026×10^1
3	5.999	299.08	49.872	0.47737×10^1

Table 3. Determination of parabolic layer constants, using 6 data points rounded to 5 km, by the method of least squares.

Number of Iterations	Parameter Values f_P	h_m	a	$\sum_{i=1}^{6} (\text{Res.})_i^2$
0	6.0000	156.00	134.00	0.20066×10^6
1	5.9999	299.09	49.873	0.48329×10^1
2	5.9999	298.89	49.869	0.48003×10^1
3	5.9999	299.08	49.873	0.47738×10^1

Table 4. Determination of parabolic layer constants, using 6 data points rounded to 5 km, by the method of least squares.

(c) Convergence was not affected by noise in the data, i.e. since our system is overdetermined the effects of noise are minimized.

(d) The method is most sensitive to the parameter f_P.

Remembering that we perform a Taylor expansion of I about the initial parameter values and drop all terms higher than first order, we are somewhat surprised that all of the sets of "guesses" converge, since in some cases the corrections were of the same order of magnitude as the oA_i. The explanation is as follows:

$$I(f_P, h_m, a, f) = I_o + \left(\frac{\partial I}{\partial f_P}\right)_o \Delta f_P + \left(\frac{\partial I}{\partial h_m}\right)_o \Delta h_m + \left(\frac{\partial I}{\partial a}\right)_o \Delta a$$

$$+ \ldots + \text{higher order terms in } \Delta f_P, \ \Delta h_m, \ \Delta a$$

Where I_o is I evaluated at the point (f_{P_o}, h_{m_o}, a_o) and the partial derivatives are similarly defined. The higher order terms may be neglected if a suitable combination of the following two conditions are satisfied.

(1) $|\Delta b_j| << M_j$, where j = 1, 2, 3 and M_j is some upper bound (letting $b_1 = f_P$, $b_2 = h_m$, and $b_3 = a$) How M_j is chosen is given by Morrison (1959).

(2) $\left(\frac{\partial^2 I}{\partial b_i \partial b_k}\right) << 1$. This condition depends on the functional form of $f_N^2(h)$. In the model studies, $f_N^2(h) = f_P^2 \left\{1 - \frac{(h-h_m)^2}{a^2}\right\}$

Then $I(f_P, h_m, a, f) = h_m - a + \int_{h_o}^{h_R} \mu' \, dh$

where $h_R = h_m - a\sqrt{1 - f^2/f_P^2}$ and $h_o = h_m - a$. Since $f_N^2(h)$ is a monotonic function of h, we can consider the inverse function.

$$I(f_P, h_m, a, f) = h_o + \int_o^{f^2} \mu'(f, f_H, f_N) \frac{dh}{df_N^2} \, df_N$$

where $h = h_m - a\sqrt{1 - f_N^2/f_P^2} \rightarrow \frac{dh}{df_N^2} = \frac{a/2}{f_P^2\sqrt{1 - f_N^2/f_P^2}}$

Then $I(f_P, h_m, a, f) = h_m - a + \frac{a}{2} \int_o^{f^2} \frac{\mu'(f, f_H, f_N^2)}{f_P^2\sqrt{1 - f_N^2/f_P^2}} \, df_N^2$

$$\frac{\partial^k I}{\partial h_m^k} = \frac{\partial^k I}{\partial a^k} = 0 \text{ for } k \geq 2 .$$

But $\frac{\partial^k I}{\partial f_P^k} > 0$ for $k \geq 2$. But our initial values of f_P, f_{P_o}, were quite close to the actual f_P, i.e. $\Delta f_P = 0$ in three tests and $\Delta f_P = 0.5$ for the fourth case. We therefore see that the parabolic model has the property that it is linear in two of the three parameters, a property which makes this functional form particularly appealing for use in a lamination method.

IV. SUMMARY AND CONCLUSIONS

The problem is to determine under what conditions equation (1.1) and equation (1.2) uniquely specify the plasma frequency as a function of height, $f_N(h)$; and, providing that these restrictions are satisfied, to determine what numerical techniques which are compatible with reasonable physical assumptions are available for inverting these equations.

For $f_{min} > 0$, neither equation alone allows a unique solution for any dip angle, θ. For $f_{min} > 0$, both equations together do not give a unique solution for the case of longitudinal propagation, $\theta = 0^o$, if the z-trace and x-trace are used. For $\theta > 0^o$ and $f_{min} > 0$, we have assumed that the solution is unique when the x-trace and o-trace are used.

For $f_{min} = 0$ or when either virtual height trace can be extrapolated to zero, we have presented an iterative method based on minimizing the sum of the squared residuals between the observed and calculated virtual heights, where the calculated virtual heights are derived from a chosen functional form. The power of this approach is that the independent variable is h, whereas in other methods the independent variable is $\varphi(f_N)$. Since h is the independent variable, we need no longer restrict ourselves to monotonic functions of h, and can easily include the variation of the gyrofrequency, f_H, with height (as we must on the topside). The weakness of this approach is that it is an iterative technique, and, consequently, is only feasible if a high speed computer is available. Even then it is slow compared to the methods of Doupnik and Paul and Wright. How to determine the initial parameter values is also a problem. Schmerling has suggested a reasonable solution which is to crudely determine $f_N(h)$ from the ionogram using some manual technique, e.g. slider method of Schmerling and Ventrice (1958).

Titheridge (1959) has suggested a solution for the "start" problem which we have extended. This solution is inherently superior to the model approaches of Paul and Wright (1963), Doupnik (1963), etc. since it does not make a model assumption for the underlying ionization.

We have developed a technique for the reduction of ionograms to electron density height profiles which is based on a least-squares method, and has many advantages over the other techniques now in use. No attempt has been made to develop this to the stage where it is suitable for the routine reduction of large numbers of ionograms, but we feel that this approach is very promising for this purpose, and suggest that further studies should be continued as follows:

(1) Model studies to refine the low-frequency solution given on page 27.

(2) Model studies of the valley problem, where both monotonic and non-monotonic functions of h are used as discussed on page 34.

(3) Model studies of the topside (above F2 peak) where the gyro-frequency variation is included as given on page 23.

(4) Model studies on the effect of the initial guesses on convergence, for models other than a parabola.

(5) Investigation of the uniqueness of the solution for $f_{min} > o$ and $\theta > o$.

(6) Investigation of more efficient programming techniques to reduce computer time.

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